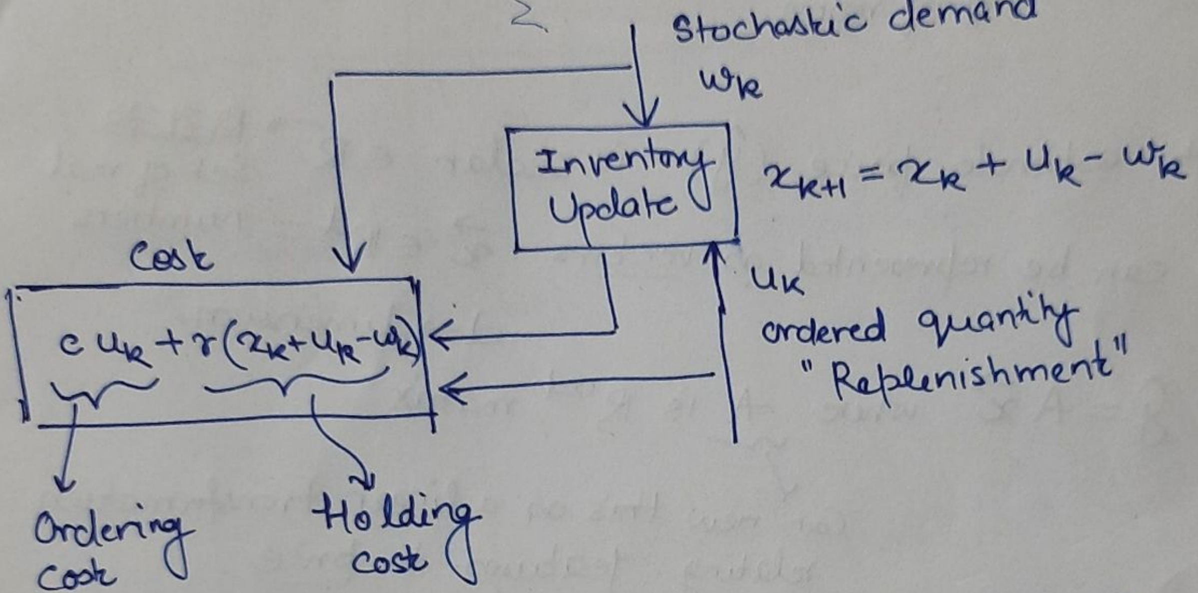
Lecture #1

Basics of Linear Algebra

Useful reference: "Essence of linear algebra"
— 3Blue1Brown.

What this course is about?

Consider a simple inventory control problem



Goal: Minimize total cost of Inventory Control

- Inventory Updates are linear
- There is an optimization problem involved
- The demand is random and follows some statistical distribution (not known exactly)

Linear Algebra
- Vectors & Matrices.
- Linear transformation
- Range & Null Spaces
- Eigenvalues & Eigenvectors
- Geometric understanding

Optimization

Statistical Tools

Python implementation

Will keep coming back to this through out the course

* Linear Algebra → Linear models are tractable

• ~~vectors~~ → Solving a system of linear equations.

Let us say we want to build a simple linear model of estimating price of a house based on certain "features" → size
location
# of rooms
⋮

Want to estimate price ($\hat{y}$). is scalar $\in \mathbb{R}$ → ~~Reals~~ Set of real numbers

Features can be represented as <u>vectors</u> $\vec{x} \in \mathbb{R}^d$

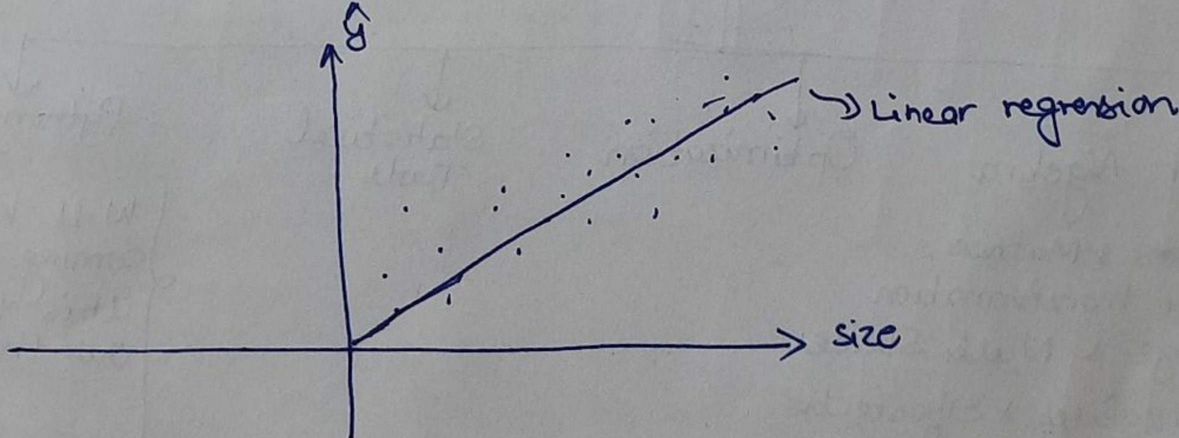
$d \rightarrow$ dimension

<u>Model</u>: $\hat{y} = A\vec{x}$ where A is $\mathbb{R}^{1\times d}$ matrix

Can view this as a linear transformation relating features to price

$$\hat{y} = \underbrace{[a_1 \ a_2 \cdots \ a_d]}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}}_{\vec{x}}$$

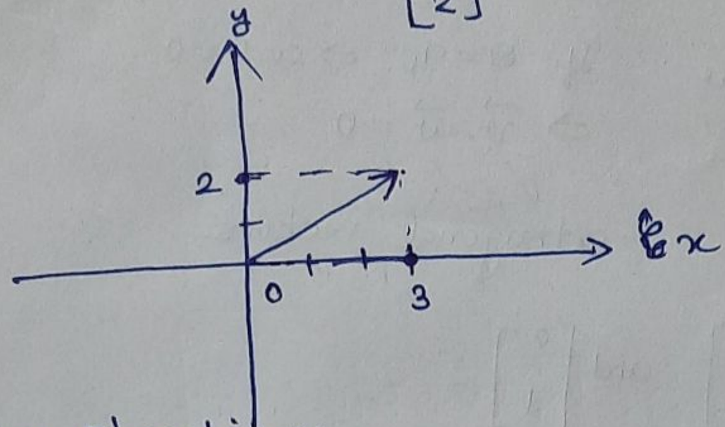
$$\hat{y} = a_1 x_1 + a_2 x_2 + \cdots + a_d x_d$$



- **Vectors**: Are specified by length and direction

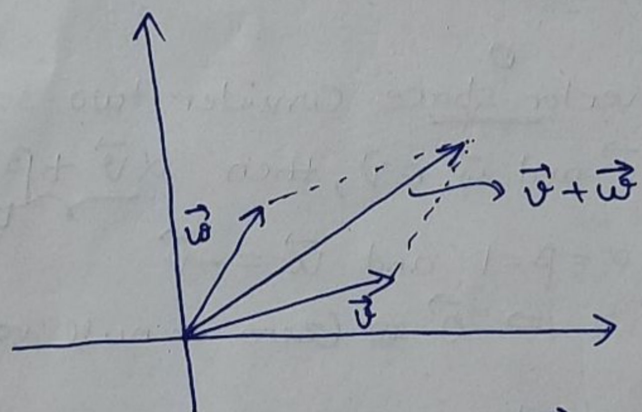
Consider the vector $\vec{v} \in \mathbb{R}^2$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

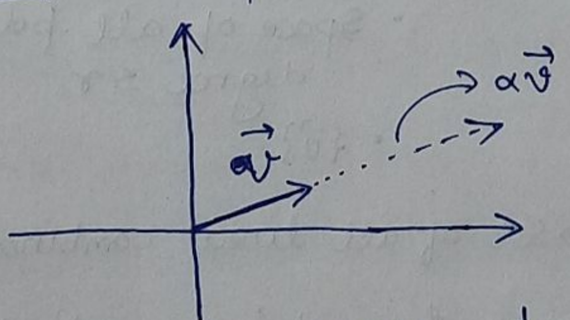


Vector operations:

↳ Vector addition



↳ Scalar multiplication: $\alpha\vec{v}$ where $\alpha \in \mathbb{R}$



↳ Dot product : $\vec{v}.\vec{w} = \sum_{i=1}^{d} v_i w_i$
or Inner

$\langle \vec{v}, \vec{w} \rangle$ or $\vec{v}^T\vec{w}$ or $\vec{w}^T\vec{v}$

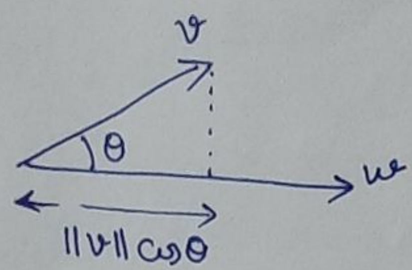
Length or magnitude of a vector:
(Norm of a vector)

- L_2-norm
or Euclidean Norm $\quad \|\vec{v}\|_2$ or simply by $\|\vec{v}\|$

$$:= \sqrt{v_1^2 + v_2^2 + \dots + v_d^2}$$

Property: $\vec{v}.\vec{v} = \|v\|^2$

* Geometric interpretation of dot product.



$\vec{v}.\vec{w} = \|v\| \, \|w\| \cos\theta$

If $\theta = 90° \Rightarrow \cos\theta = 0$

$\Rightarrow \vec{v}.\vec{w} = 0$

orthogonal vectors.

Example: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

* Vector Space: Vectors reside in Vector space $\mathcal{V}$

Properties of vector space: Consider two scalars $\alpha, \beta \in \mathbb{R}$ and vectors $\vec{v}$ and $\vec{w} \in \mathcal{V}$, then $\underbrace{\alpha\vec{v} + \beta\vec{w}}_{\text{Linear Combination of } \vec{v} \text{ and } \vec{w}} \in \mathcal{V}$

Choose $\alpha = \beta = 1$ and $\vec{w} = -\vec{v}$

$\Rightarrow \vec{0}$ (zero or null vector) $\in \mathcal{V}$

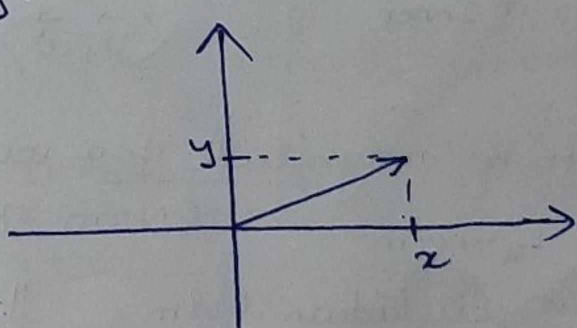
Example of vector spaces:
- Euclidean space $\mathbb{R}^d$
- Space of all polynomials of degree $\leq r$
- $\{\vec{0}\}$

* Span of vectors: → set of all linear combinations.

Vector space formed by a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\hat{i} + y\hat{j}$$

$$= x\begin{bmatrix} 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$$

* Basis Vectors:

↳ Linearly dependent and linearly independent vectors

$\begin{bmatrix}1\\-3\end{bmatrix}$ and $\begin{bmatrix}-2\\6\end{bmatrix}$

$\underline{\underline{\vec{v}}}$ $\quad$ $\underline{\underline{\vec{w}}}$

$\vec{w} = -2\vec{v}$

↳ Linearly dependent

$\alpha\vec{v} + \beta\vec{w} = \alpha\vec{v} - 2\beta\vec{v}$

$= (\alpha - 2\beta)\vec{v}$

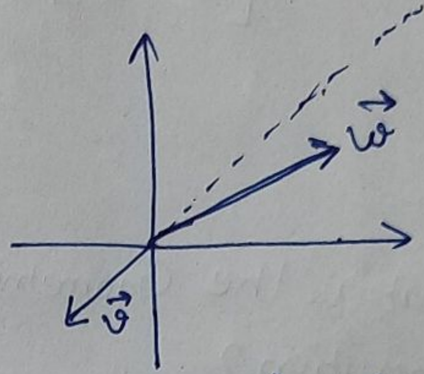
Span $\left\{ \begin{bmatrix}1\\-3\end{bmatrix}, \begin{bmatrix}2\\-6\end{bmatrix} \right\}$ is simply a scalar multiple of one of the vectors.

$\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$

$\vec{v}$ $\quad$ $\vec{w}$

There does not exist any α such that

$\vec{v} = \alpha\vec{w}$



$\therefore$ $\alpha\vec{v} + \beta\vec{w} = \vec{0}$ if and only if $\alpha = \beta = 0$.

Such vectors are called linearly independent vectors.

<u>Basis</u>: The basis of a vector space is a set of linearly independent vectors that span the full space.

<u>Q</u>: Is basis unique? No

In $\mathbb{R}^2$, the standard (normal) basis vectors are

$\hat{i} = \begin{bmatrix}1\\0\end{bmatrix}, \hat{j} = \begin{bmatrix}0\\1\end{bmatrix}$

$\therefore \begin{bmatrix}x\\y\end{bmatrix} = x\hat{i} + y\hat{j} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$

Consider another set of vectors $\vec{v} = \begin{bmatrix}1\\-1\end{bmatrix}$ $\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}$

then $\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\-1\end{bmatrix} + (x+y)\begin{bmatrix}0\\1\end{bmatrix}$

$= x\vec{v} + (x+y)\vec{w} \Rightarrow$ Basis vectors

* Linear transformations (Origin of Matrices)

Consider a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

What is $A\vec{v} = ?$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

$$= x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

- What is the Geometric intuition behind this operation?
- What is linear transformation?
  ↳ Transformation is any vector valued function.

$$L(\vec{v}) = \vec{w}$$

↳ L is transformation.

If transformation is linear, then.

$$L(\alpha\vec{v}_1 + \beta\vec{v}_2) = \alpha L(\vec{v}_1) + \beta L(\vec{v}_2)$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix}$$

$$\Rightarrow \left.\begin{matrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} a \\ c \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} b \\ d \end{bmatrix} \end{matrix}\right\}$$ Columns of A indicate corresponding transformations.